6.1 OPTIMUM RECEIVERS FOR SIGNALS CORRUPTED BY AWGN

Let us begin by developing a mathematical model for the signal at the input to the receiver. We assume that the transmitter sends digital information by use of M signal waveforms $\{s_m(t),\ m = 1,2, \ldots, M\}$. Each waveform is transmitted within the symbol (signaling) interval of duration T. To be specific, we consider the transmission of information over the interval $0 \leq t \leq T$.

The channel is assumed to corrupt the signal by the addition of white Gaussian noise as illustrated in Figure 6.1.

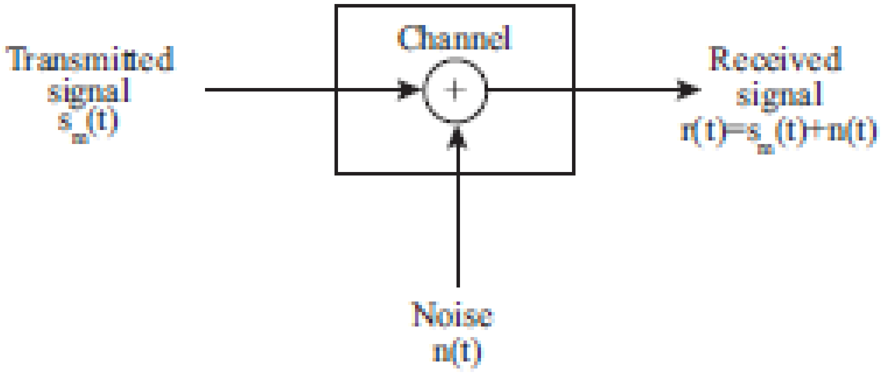


Figure 6.1 Model for received signal passed through an AWGN channel

Thus, the received signal in the interval $0 \le t \le T$ may be expressed as

$$r(t) = s_m(t) + n(t) \tag{6.1}$$

with power spectral densisty of $n(t)$

$$\Phi_{nn}(f) = \frac{1}{2} N_0 \; [W/Hz] \tag{6.2}$$

Based on the observation of $r(t)$ over the signal interval, we wish to design a receiver that is optimum in the sense that it minimizes the probability of making an error.

It is convenient to subdivide the receiver into two parts – the signal *demodulator* and the *detector* – Figure 6.2.

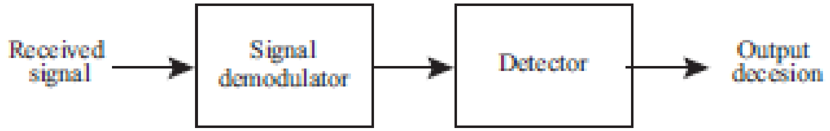


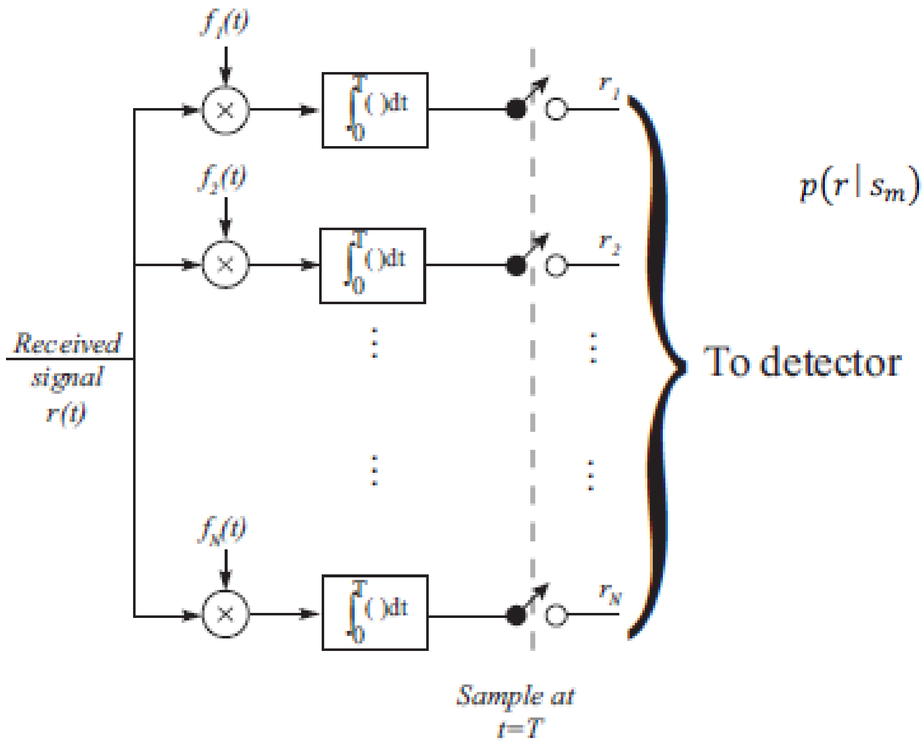
Figure 6.2 Receiver configuration

The function of the signal demodulator is to convert the received waveform $r(t)$ into N-dimensional vector $r = [r_1 r_2 \dots r_N]$, where N is the dimension of the transmitted signal waveforms. The function of the detector is to decide which of M possible signal waveforms was transmitted based of the vector r.

Two realizations of the signal demodulator are described in the next two sections. One is based on the used of *signal correlators*. The second is based on the use of *matched filters*. The optimum detector that follows the signal demodulator is designed to *minimize the probability of error*.

6.1.1 *Correlation demodulator*

We describe a correlation demodulator that decomposes the received signal and noise into N-dimensional vectors. The signal and the noise are expanded into a series of linearly weighted orthonormal basis functions $\{f_n(t)\}$. It is assumed that the N basis functions $\{f_n(t)\}$ span the signal space, so that every one of the possible transmitted signal of the set $\{s_m(t)\}$, can be represented as a linear combination of $\{f_n(t)\}$. In the case of the noise, the functions $\{f_n(t)\}$ do not span the noise space. However, the noise terms that fall outside the signal space are irrelevant to the detection of the signal. Suppose the received signal $r(t)$ is passed through a parallel bank of N cross correlators which basically compute the projection of $r(t)$ onto the N basis functions $\{f_n(t)\}$, as illustrated in Figure 6.3.



$$p(r\,|\,s_m) = \frac{1}{(\pi N_0)^{\frac{N}{2}}}\exp\left[-\sum_{k=1}^{N}\frac{(r_k - s_{mk})^2}{N_0}\right], m = 1, 2, \ldots, M$$

Figure 6.3 Correlation-type demodulator

6.1.2 Matched-Filter demodulator

Instead of using a bank of N correlators to generate the variables $\{r_k\}$, we may use a bank of N linear filters. Suppose that the impulse responses of the N filters are

$$h_k(t) = f_k(T - t), 0 \le t \le T \tag{6.11}$$

where $\{f_k(t)\}$ are the N basis functions and $h_k(t) = 0$ outside of the interval $0 \le t \le T$. The outputs of these filters are

$$\begin{aligned} y_k(t) &= \int_0^t r(\tau) h_k(t - \tau) d\tau \\ &= \int_0^t r(\tau) f_k(T - t + \tau) d\tau , 0 \le t \le T \end{aligned} \tag{6.12}$$

Now, if we sample the outputs of the filters at $t = T$, we obtain

$$y_k(T) = \int_0^T r(\tau) f_k(\tau)\, d\tau = r_k \tag{6.13}$$

Hence, the sampled outputs of the filters at time $t = T$ are exactly the set of values $\{r_k\}$ obtained from the N linear correlators.

A filter whose impulse response $h(t) = s(T-t)$ is called ***matched filter*** to the signal $s(t)$.

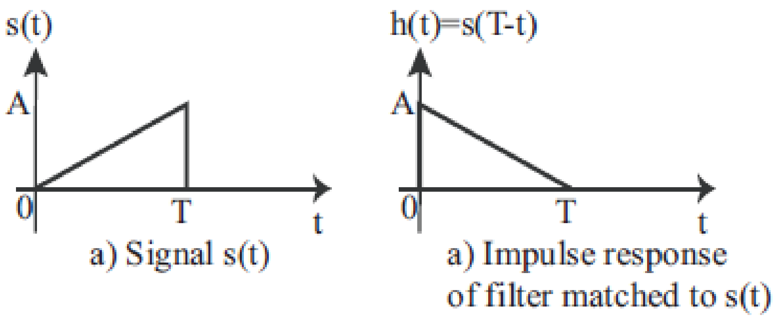


Figure 6.4 Signal s(t) and filter matched to s(t)

Properties of the matched filter:

- If a signal s(t) is corrupted by AWGN, the filter with an impulse response matched to s(t) maximizes the output signal-to-noise ratio (SNR), which is as follows

$$SNR_0 = \frac{2}{N_0}\int_0^T s^2(t)\,dt = \frac{2E}{N_0} \qquad (6.15)$$

- Note that the output SNR from the matched filter depends on the energy of the waveform $s(t)$ but not on the detailed characteristics of $s(t)$. This is another interesting property of the matched filter.

In the case of the demodulator described above, the N matched filters are matched to the basis functions $\{f_k(t)\}$. Figure 6.6 illustrates the matched filter demodulator that generates the observed variables $\{r_k\}$.

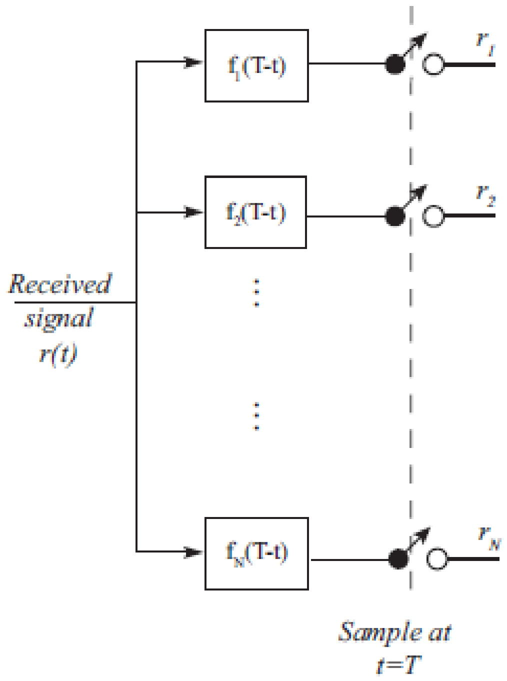
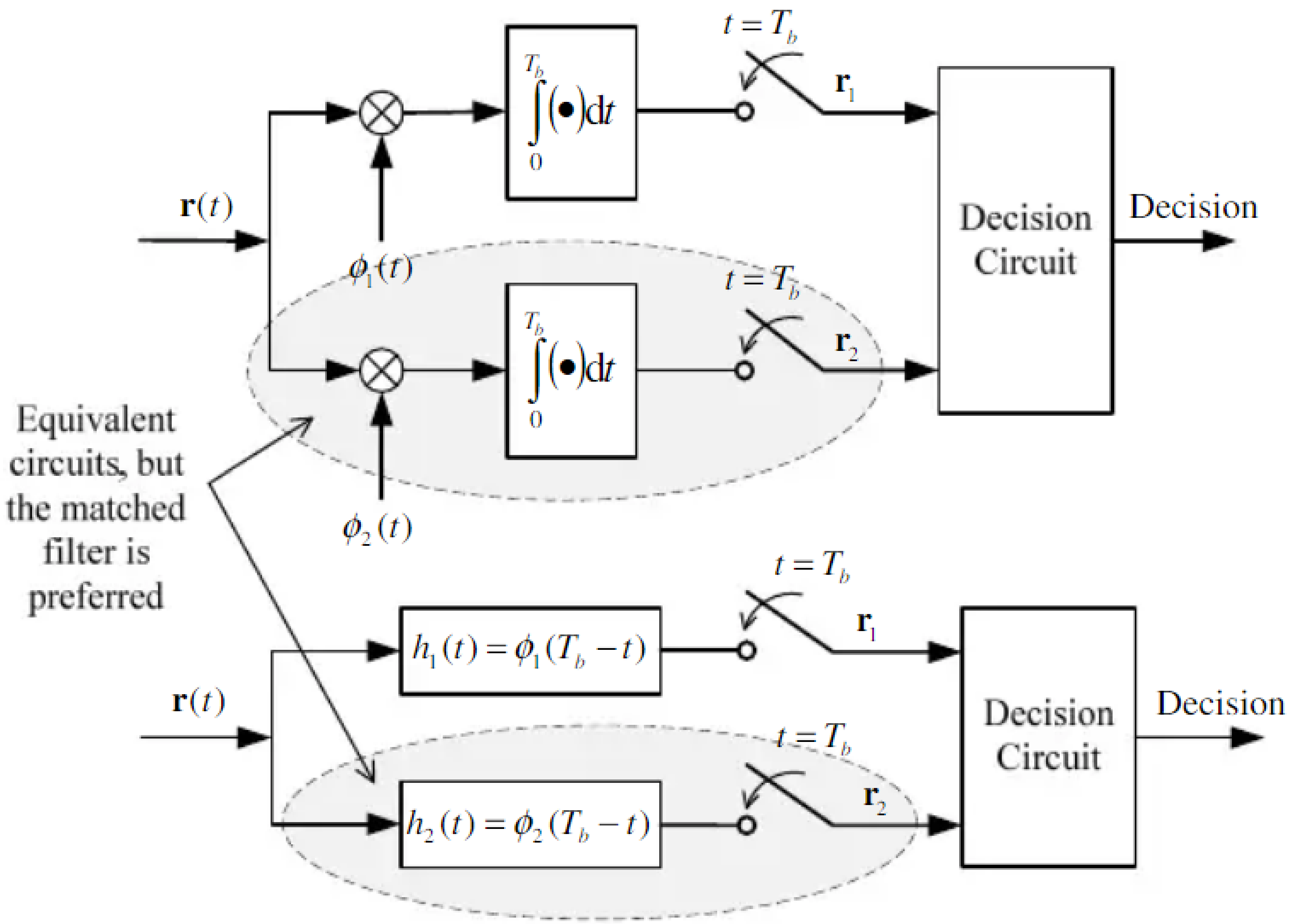


Figure 6.6 Matched filter demodulator



$t = T_b$
$\mathbf{r}_1$
$\int_0^{T_b} (\bullet) dt$
$\mathbf{r}(t)$
Decision Circuit
Decision
$\phi_1(t)$
$t = T_b$
$\mathbf{r}_2$
$\int_0^{T_b} (\bullet) dt$
Equivalent circuits, but the matched filter is preferred
$\phi_2(t)$
$t = T_b$
$\mathbf{r}_1$
$h_1(t) = \phi_1(T_b - t)$
$\mathbf{r}(t)$
Decision Circuit
Decision
$t = T_b$
$\mathbf{r}_2$
$h_2(t) = \phi_2(T_b - t)$

6.1.3 The Optimum detector

We have demonstrated that, for a signal transmitted over an AWGN channel, either a correlation demodulator or a matched filter demodulator produces the vector $r = [r_1 r_2 \ldots r_N]$, which contains all the relevant information in the received signal waveform. In this section we describe the optimum decision rule based on the observation vector r. We assume that there is no memory in signals transmitted in successive signal intervals.

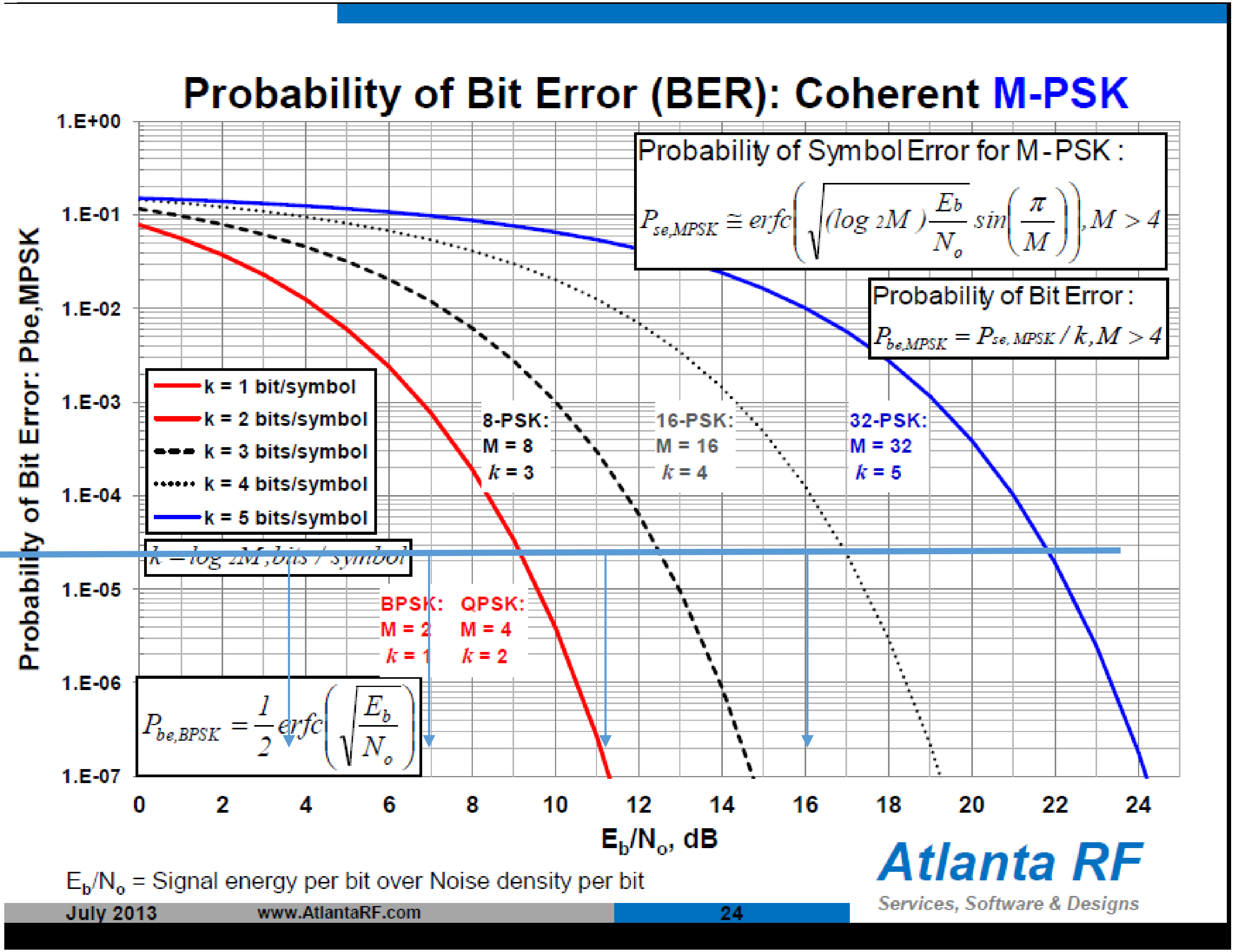
We wish to design a signal detector that makes a decision on the transmitted signal in each signal interval based on the observation of the vector r in each interval such that the probability of a correct decision is maximized.

Euclidean distance (minimum distance detection)

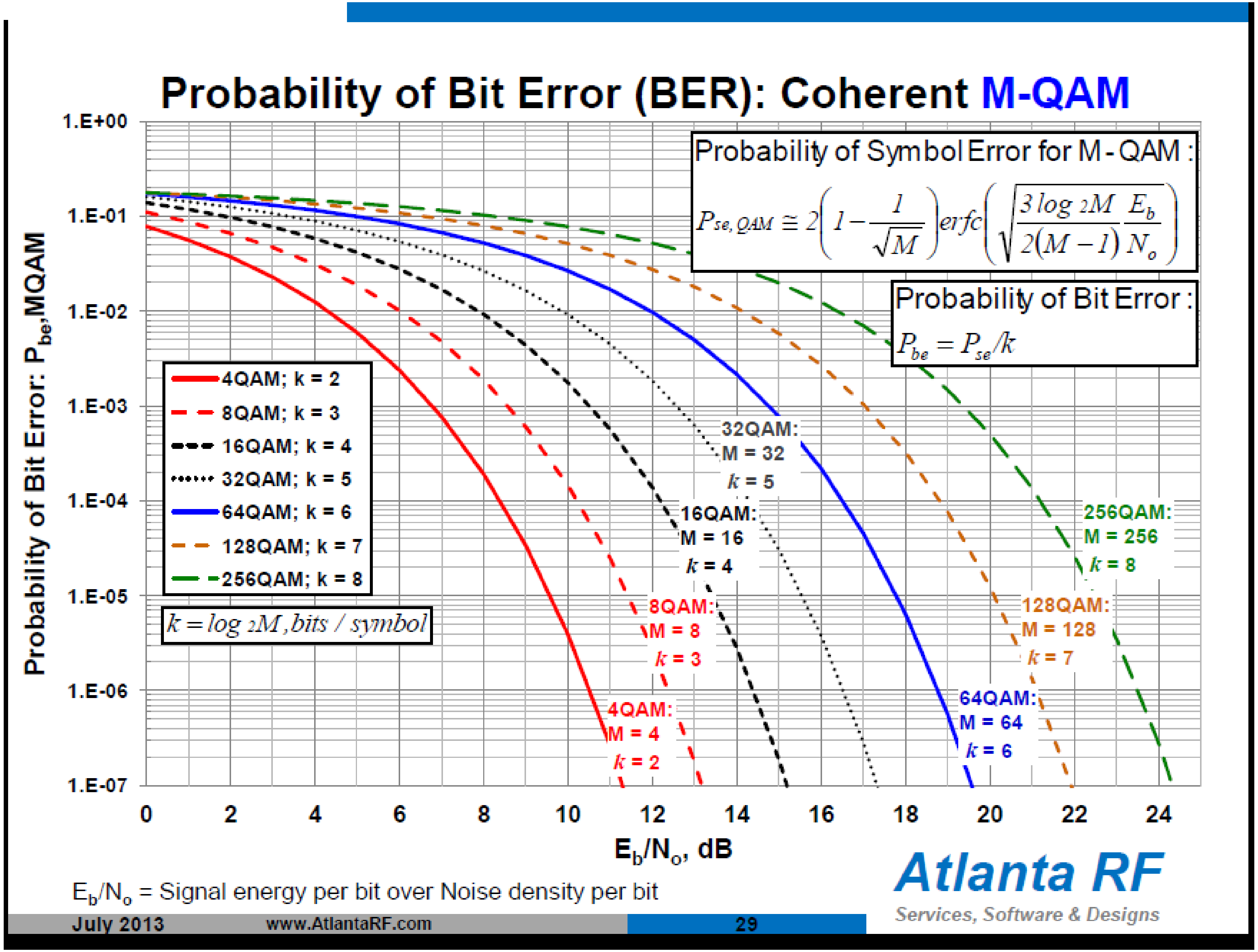
$$D(r, s_m) = \sum_{k=1}^{N} (r_k - s_{mk})^2 \qquad (6.20)$$

We call $D(r, s_m), m = 1, 2, \ldots, M$, the distance metrics. Hence, for the AWGN channel, the decision rule based on the ML criterion reduces to finding the signal s_m that is closest in distance to the received signal vector r. We shall refer to this decision rule as minimum distance detection.

$$P_{\mathrm{BER-MPSK}} \approx \frac{2}{\log_2 M} Q\left(\sqrt{\frac{2E_b \log_2 M}{N_0}} \sin\left(\frac{\pi}{M}\right)\right)$$



$$P_{\mathrm{BER-QAM}} \approx \left(\frac{4(\sqrt{M}-1)}{\sqrt{M}\log_2 M}\right) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{E_b}{N_0}}\right)$$



6.2.1 Probability of Error for Binary Modulation

PAM (antipodal)

Let us consider binary PAM signals, where the two signal waveforms are $s_1(t) = g(t)$ and $s_2(t) = -g(t)$, and $g(t)$ is an arbitrary pulse that is nonzero in the interval $0 \le t \le T_b$ and zero elsewhere. The energy in the pulse $g(t)$ is E_g.

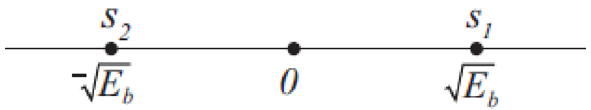


Figure 6.10 Signal points for antipodal signal.

Let us assume that the two signals are equally likely and that signal $s_1(t)$ was transmitted. Then the received signal from the (matched filter or correlation) demodulator is

$$r = s_1 + n = \sqrt{E_b} + n \tag{6.33}$$

where n represents the additive Gaussian noise component, which has zero mean and variance $\sigma_n^2 = \frac{1}{2}N_0$. In this case, the decision rule based on the correlation metric given by Equation 6.23 compares r with the threshold zero. If $r > 0$, the decision is made favor of $s_1(t)$, and if $r < 0$, the decision is made that $s_2(t)$, was transmitted. The two conditional PDFs of r are

$$p(r \mid s_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r-\sqrt{E_b})^2}{N_0}\right] \tag{6.34}$$

$$p(r \mid s_2) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r+\sqrt{E_b})^2}{N_0}\right] \tag{6.35}$$

These two conditional PDFs are shown in Figure 6.11.

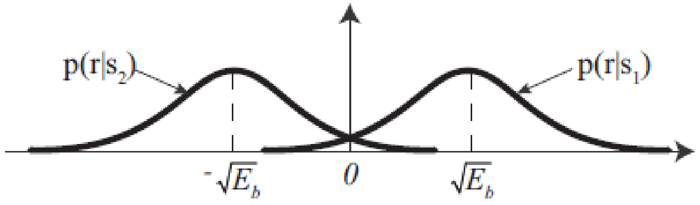


Figure 6.11 Conditional PDF's of two signals.

Given that $s_1(t)$, was transmitted, the probability of error is simply the probability that $r < 0$, i.e.,

$$P(e \mid s_1) = \int_{-\infty}^{0} p(r \mid s_1)\, dr = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{6.36}$$

Where $Q(x)$ is the Q-function. Similarly, if we assume that $s_2(t)$, was transmitted, $r = -\sqrt{E_b} + n$ and the probability that $r > 0$ is also $P(e \mid s_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$.

Since the signals $s_1(t)$ and $s_2(t)$are equally likely to be transmitted, the average probability of error is

$$P_b = \frac{1}{2}P(e \mid s_1) + \frac{1}{2}P(e \mid s_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{6.37}$$

The probability may be also expressed in terms of the distance between the two signals s_1 and s_2. From Figure 6.10, we observe that the two signals are separated by the distance $d_{12} = 2\sqrt{E_b}$. By substituting $E_b = \frac{1}{4}d_{12}^2$ into Equation 6.37, we obtain

$$P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) \tag{6.38}$$

Gaussian (normal) distribution

The PDF is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-m_x}{\sigma}\right)^2\right] \tag{3.40}$$

Where m_x is the mean and σ^2 is the variance of the random variable.

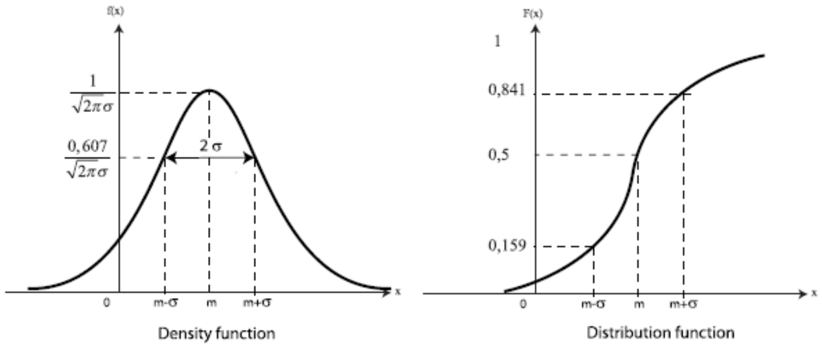


Figure 3.5 The PDF and CDF of a Gaussian random variable

Binary orthogonal signals

Signal vectorss_1 and $s_2$are two-dimensional, as shown in Figure 6.12, and may be expressed as

$$s_1 = [\sqrt{E_b}, 0]$$
$$s_2 = [0, \sqrt{E_b}] \tag{6.39}$$

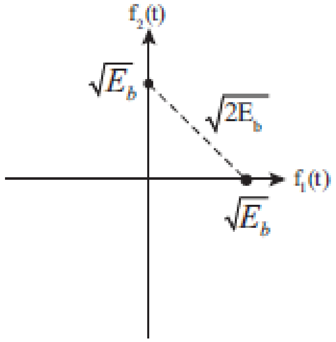


Figure 6.12 Signal points for binary orthogonal signals.

The average error probability for binary orthogonal signals is

$$P_b = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q(\sqrt{\gamma_b})$$

where by definition, γ_b is the SNR per bit.

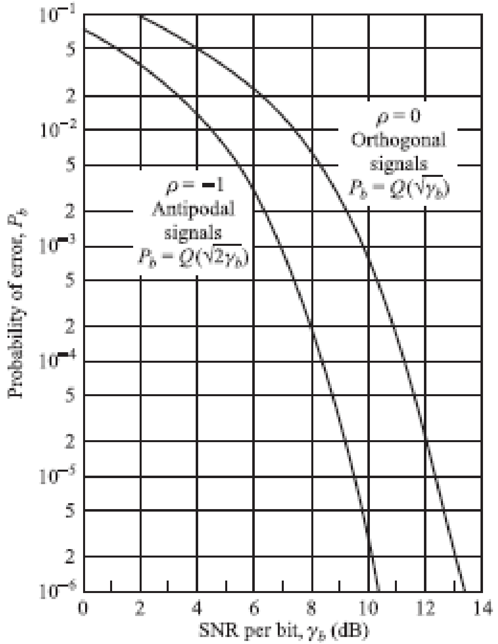


Figure 6.13 Error probability for binary antipodal and binary orthogonal signaling.

$$6\,[dB] = 10 \log \frac{E_b}{N_0} \; ; \; 10^{0,6} = \frac{E_b}{N_0} = 3{,}98$$

$$\frac{2E_b}{N_0} = 7{,}96$$

$$P_e = Q\left(\sqrt{7{,}96}\right) = 2{,}4 \cdot 10^{-3} \text{ – antipodal}$$

$$P_e = Q\left(\sqrt{3{,}98}\right) = 2{,}3 \cdot 10^{-2} \text{ – orthogonal}$$

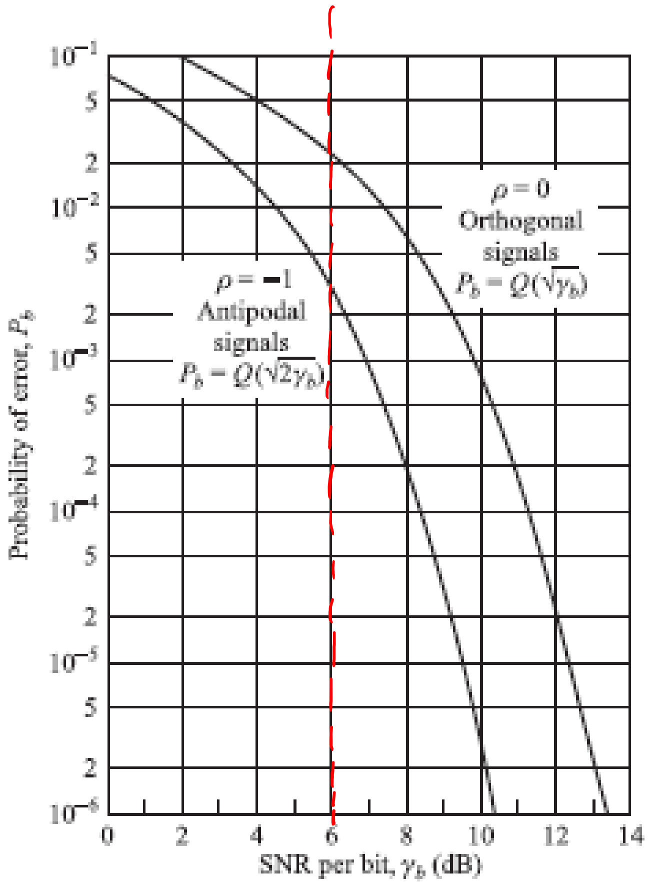


Figure 6.13 Error probability for binary antipodal and binary orthogonal signaling.

$$8\,[dB] = 10 \log E_b/N_0$$

$$10^{0,8} = \frac{E_b}{N_0} = 6,3$$

$$\frac{2E_b}{N_0} = 12,6$$

$$P_e = Q\left(\sqrt{12,6}\right) = Q\big(\underbrace{3,55}_{x}\big) \doteq 1,9 \cdot 10^{-4}$$

$$4\,[\mathrm{dB}] = 10 \log \frac{E_b}{N_0}$$

$$10^{0,4} = \frac{E_b}{N_0} \doteq 2,512$$

$$\frac{2E_b}{N_0} = 5,024$$

$$P_e = Q\left(\sqrt{5,024}\right) = Q\left(2,24\right) \doteq 1.2 \cdot 10^{-2}$$

Table 1: Values of $Q(x)$ for $0 \leq x \leq 9$

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.00	0.5	2.30	0.010724	4.55	2.6823×10^{-6}	6.80	5.231×10^{-12}
0.05	0.48006	2.35	0.0093867	4.60	2.1125×10^{-6}	6.85	3.6925×10^{-12}
0.10	0.46017	2.40	0.0081975	4.65	1.6597×10^{-6}	6.90	2.6001×10^{-12}
0.15	0.44038	2.45	0.0071428	4.70	1.3008×10^{-6}	6.95	1.8264×10^{-12}
0.20	0.42074	2.50	0.0062097	4.75	1.0171×10^{-6}	7.00	1.2798×10^{-12}
0.25	0.40129	2.55	0.0053861	4.80	7.9333×10^{-7}	7.05	8.9459×10^{-13}
0.30	0.38209	2.60	0.0046612	4.85	6.1731×10^{-7}	7.10	6.2378×10^{-13}
0.35	0.36317	2.65	0.0040246	4.90	4.7918×10^{-7}	7.15	4.3389×10^{-13}
0.40	0.34458	2.70	0.003467	4.95	3.7107×10^{-7}	7.20	3.0106×10^{-13}
0.45	0.32636	2.75	0.0029798	5.00	2.8665×10^{-7}	7.25	2.0839×10^{-13}
0.50	0.30854	2.80	0.0025551	5.05	2.2091×10^{-7}	7.30	1.4388×10^{-13}
0.55	0.29116	2.85	0.002186	5.10	1.6983×10^{-7}	7.35	9.9103×10^{-14}
0.60	0.27425	2.90	0.0018658	5.15	1.3024×10^{-7}	7.40	6.8092×10^{-14}
0.65	0.25785	2.95	0.0015889	5.20	9.9644×10^{-8}	7.45	4.667×10^{-14}
0.70	0.24196	3.00	0.0013499	5.25	7.605×10^{-8}	7.50	3.1909×10^{-14}
0.75	0.22663	3.05	0.0011442	5.30	5.7901×10^{-8}	7.55	2.1763×10^{-14}
0.80	0.21186	3.10	0.0009676	5.35	4.3977×10^{-8}	7.60	1.4807×10^{-14}
0.85	0.19766	3.15	0.00081635	5.40	3.332×10^{-8}	7.65	1.0049×10^{-14}
0.90	0.18406	3.20	0.00068714	5.45	2.5185×10^{-8}	7.70	6.8033×10^{-15}
0.95	0.17106	3.25	0.00057703	5.50	1.899×10^{-8}	7.75	4.5946×10^{-15}
1.00	0.15866	3.30	0.00048342	5.55	1.4283×10^{-8}	7.80	3.0954×10^{-15}
1.05	0.14686	3.35	0.00040406	5.60	1.0718×10^{-8}	7.85	2.0802×10^{-15}
1.10	0.13567	3.40	0.00033693	5.65	8.0224×10^{-9}	7.90	1.3945×10^{-15}
1.15	0.12507	3.45	0.00028029	5.70	5.9904×10^{-9}	7.95	9.3256×10^{-16}
1.20	0.11507	3.50	0.00023263	5.75	4.4622×10^{-9}	8.00	6.221×10^{-16}
1.25	0.10565	3.55	0.00019262	5.80	3.3157×10^{-9}	8.05	4.1397×10^{-16}
1.30	0.0968	3.60	0.00015911	5.85	2.4579×10^{-9}	8.10	2.748×10^{-16}
1.35	0.088508	3.65	0.00013112	5.90	1.8175×10^{-9}	8.15	1.8196×10^{-16}
1.40	0.080757	3.70	0.0001078	5.95	1.3407×10^{-9}	8.20	1.2019×10^{-16}
1.45	0.073529	3.75	8.8417×10^{-5}	6.00	9.8659×10^{-10}	8.25	7.9197×10^{-17}
1.50	0.066807	3.80	7.2348×10^{-5}	6.05	7.2423×10^{-10}	8.30	5.2056×10^{-17}
1.55	0.060571	3.85	5.9059×10^{-5}	6.10	5.3034×10^{-10}	8.35	3.4131×10^{-17}
1.60	0.054799	3.90	4.8096×10^{-5}	6.15	3.8741×10^{-10}	8.40	2.2324×10^{-17}
1.65	0.049471	3.95	3.9076×10^{-5}	6.20	2.8232×10^{-10}	8.45	1.4565×10^{-17}
1.70	0.044565	4.00	3.1671×10^{-5}	6.25	2.0523×10^{-10}	8.50	9.4795×10^{-18}
1.75	0.040059	4.05	2.5609×10^{-5}	6.30	1.4882×10^{-10}	8.55	6.1544×10^{-18}
1.80	0.03593	4.10	2.0658×10^{-5}	6.35	1.0766×10^{-10}	8.60	3.9858×10^{-18}
1.85	0.032157	4.15	1.6624×10^{-5}	6.40	7.7688×10^{-11}	8.65	2.575×10^{-18}
1.90	0.028717	4.20	1.3346×10^{-5}	6.45	5.5925×10^{-11}	8.70	1.6594×10^{-18}
1.95	0.025588	4.25	1.0689×10^{-5}	6.50	4.016×10^{-11}	8.75	1.0668×10^{-18}
2.00	0.02275	4.30	8.5399×10^{-6}	6.55	2.8769×10^{-11}	8.80	6.8408×10^{-19}
2.05	0.020182	4.35	6.8069×10^{-6}	6.60	2.0558×10^{-11}	8.85	4.376×10^{-19}
2.10	0.017864	4.40	5.4125×10^{-6}	6.65	1.4655×10^{-11}	8.90	2.7923×10^{-19}
2.15	0.015778	4.45	4.2935×10^{-6}	6.70	1.0421×10^{-11}	8.95	1.7774×10^{-19}
2.20	0.013903	4.50	3.3977×10^{-6}	6.75	7.3923×10^{-12}	9.00	1.1286×10^{-19}
2.25	0.012224						